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Fractional integrodifferential boundary control of the Euler-Bernoulli beam

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Abstract

Absorbing boundary conditions are generally associated to long-range memory behaviors. In the case of the Euler-Bernoulli beam, they are naturally based on Abel-Volterra operators of order 1/2. Diffusive realizations of them are introduced and used for the construction of an original and efficient boundary dynamic feedback control.

1 Introduction

In the context of force and torque boundary control of the Euler-Bernoulli beam, we consider wave absorbing feedbacks based upon the reduction of reflected waves.

Due to the specific propagative properties of the beam, such feedbacks involve fractional integrator and derivator of order 1/2 which make the closed-loop system non standard with long range memory convolution operators. The analysis and the numerical approximation of the system under consideration are considerably simplified by using diffusive input-output realizations of Abel fractional integral operators which allows the representation of the closed-loop system under the traditional form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mathcal{A}X, \tag{1}$$

with $\mathcal{A}$ the infinitesimal generator of a semigroup on a convenient Hilbert space.

The well-posedness and the asymptotic stability of the system are then based upon classical semigroup theory. The numerical approximation of (1) leads to a simple finite-dimensional version of this system and significant simulations are presented as illustrative examples.

2 Wave-absorbing feedback controls

We consider the problem of stabilization of the Euler-Bernoulli beam, under boundary conditions and boundary control, written as :

$$\begin{cases} \partial_t^2\theta + \partial_x^4\theta = 0, \; x \in (0,1) \\ \theta(x,0) = \theta_0(x) \\ \partial_t\theta(x,0) = \theta_1(x) \\ \theta(0,t) = 0 \\ \partial_x\theta(0,t) = 0 \\ \partial_x^2\theta(1,t) = u(t) \\ \partial_x^3\theta(1,t) = v(t), \end{cases} \tag{2}$$

with inputs u and v, respectively torque and force, and outputs y and z, respectively the deflection and slope velocities at the end of the beam :

$$\begin{cases} y(t) := \partial_t\theta(1,t) \\ z(t) := \partial_t\partial_x\theta(1,t). \end{cases} \tag{3}$$

2.1 Feedback controls under consideration

We consider feedback controls : $(u,v)^T = \mathcal{Z}(y,z)^T$, corresponding to (boundary) conditions of frequency-independent absorption of waves at $x = 1$. Such controls were introduced and investigated in [7]. The correspondence $\mathcal{Z}$ involves Abel integrators and derivators of order 1/2. The present study is devoted to the construction of diffusive realizations of such operators and then to the analysis and numerical simulation of the resulting systems.

Boundary conditions at $x = 1$ under consideration are written :

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$$\begin{pmatrix} \partial_x^2\theta(1,t) \\ \partial_x^3\theta(1,t) \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{pmatrix} \partial_t\theta(1,t) \\ \partial_t\partial_x\theta(1,t) \end{pmatrix}, \quad (4)$$

where A, B, C, D are convenient causal convolutive operators to be specified. On the other hand, the scattering matrix at point $x = 1$:

$$\mathcal{R} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}, \quad (5)$$

is constituted by the (complex) reflection coefficients which relate the incoming and the reflected waves at $x = 1$, according to the two types of wave-modes solutions [7], respectively travelling waves and near-field waves :

- $\theta = e^{i\omega t \pm i\sqrt{\omega}x}$,
- $\theta = e^{i\omega t \pm \sqrt{\omega}x}$.

The objective is to determine A, B, C, D, such that $\alpha, \beta, \gamma, \delta$ be small and independent of the pulsation ω of the wave-mode. We have the following result [4] :

Theorem 2.1 *The scattering matrix $\mathcal{R}$ is independent of the pulsation ω if and only if the feedback matrix is of the form :*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & \dfrac{b}{\sqrt{i\omega}} \\ c\sqrt{i\omega} & d \end{bmatrix}, \quad (6)$$

with $a, b, c, d \in \mathbf{C}$; furthermore, $\mathcal{R} = 0$ if and only if :

$$\begin{cases} a = -1 \\ b = -\sqrt{2} \\ c = \sqrt{2} \\ d = 1. \end{cases} \quad (7)$$

Remark : in the case $\mathcal{R} = 0$, the feedback control is non-reflecting : it realizes the virtual continuation of the beam to the semi-infinite domain $(1, +\infty)$ (also called "impedance matching" condition), so that the mechanical energy will be efficiently absorbed from the propagation properties.

The considered feedback controls will essentially be of the form (6), with the necessary additional hypothesis of asymptotic stability of the corresponding closed-loop system. Applying the inverse Laplace-transform to (6), we obtain the general expression in the time-domain :

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} a & bI^{1/2} \\ c\partial_t^{1/2} & d \end{bmatrix} \cdot \begin{pmatrix} y \\ z \end{pmatrix}, \quad (8)$$

with the fractional integrodifferential operators $I^{1/2}$, $\partial_t^{1/2}$ defined by the classical Riemann-Liouville formulas [5] :

$$I^{1/2}f(t) = \tfrac{1}{\sqrt{\pi}} \int_0^t \frac{f(s)}{\sqrt{t-s}} \mathrm{d}s, \qquad \partial_t^{1/2}f(t) = \partial_t \left(I^{1/2}f(t)\right). \quad (9)$$

It is additionally assumed that $a, b, c, d \in \mathbf{R}$, which leads to real and causal feedback controls.

The fractional operators $I^{1/2}$, $\partial_t^{1/2}$ generate several difficulties, both for analysis and numerical approximation. We propose diffusive realizations of such feedback controls, elaborated from the input-output behavior of convenient diffusion equations.

2.2 Diffusive representation of the feedback

We consider the monodimensional diffusion equation with input $f(t)$ and output $g(t)$ (∂ denoting a partial derivative) :

$$\begin{cases} \partial_t\Phi - \partial_x^2\Phi = \sqrt{2}\delta \otimes f, \ \Phi(x,0) = 0, \ x \in \mathbf{R} \\ g(t) := \sqrt{2}\Phi(0,t), \end{cases}$$

which is equivalent from Fourier-transform with respect to x ($\varphi = \mathcal{F}\Phi = \int e^{-2i\pi\xi x}\Phi \mathrm{d}x$), to :

$$\begin{cases} \partial_t\varphi + 4\pi^2\xi^2\varphi = \sqrt{2}f, \ \varphi(\xi,0) = 0, \ \xi \in \mathbf{R} \\ g(t) := \sqrt{2}\displaystyle\int_{-\infty}^{+\infty} \varphi(\xi,t)\mathrm{d}\xi; \end{cases} \quad (10)$$

Lemma 2.2 *[3] The input-output relation for (10) is: $g = I^{1/2}f$.*

Similarly, we consider now the input-output diffusion equation :

$$\begin{cases} \partial_t\psi + 4\pi^2\xi^2\psi = \sqrt{2}.2\pi\,|\xi|\,f, \ \psi(\xi,0) = 0, \quad \xi \in \mathbf{R} \\ g(t) := \sqrt{2}\displaystyle\int_{-\infty}^{+\infty} \left(-2\pi\,|\xi|\,\psi + \sqrt{2}f\right) \mathrm{d}\xi; \end{cases} \quad (11)$$

Lemma 2.3 *[4] The input-output relation for (11) is: $g = \partial_t^{1/2}f$.*

The proof is performed by considering in (10), the new output : $g := \sqrt{2}\partial_t \int \varphi \mathrm{d}\xi$ and the convenient change of function : $\psi = 2\pi\,|\xi|\,\varphi$. Thus, the systems (10) and (11) make possible the computation of $I^{1/2}f$ and $\partial_t^{1/2}f$ under standard input-output forms :

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t} = \mathcal{A}X + \mathcal{B}f \\ g = \mathcal{B}^*X, \end{cases} \text{and} \begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t} = \mathcal{A}X + \mathcal{B}f \\ g = \mathcal{B}^*(-X + \mathcal{D}f), \end{cases} \quad (12)$$

respectively. We may conclude :

Theorem 2.4 *The system (2) with the feedback control defined by (8) :*

$$\begin{cases} \partial_t^2\theta + \partial_x^4\theta = 0, \\ \partial_x^2\theta(1,t) = a\partial_t\theta(1,t) + bI^{1/2}\partial_t\partial_x\theta(1,t) \\ \partial_x^3\theta(1,t) = c\partial_t^{1/2}\partial_t\theta(1,t) + d\partial_t\partial_x\theta(1,t), \end{cases} \quad (13)$$

may be transformed into the augmented system :

$$\begin{cases} \partial_t^2\theta + \partial_x^4\theta = 0, \\ \partial_t\varphi + 4\pi^2\xi^2\varphi = \sqrt{2}\partial_t\partial_x\theta(1,t), \\ \partial_t\psi + 4\pi^2\xi^2\psi = \sqrt{2}.2\pi\,|\xi|\,\partial_t\theta(1,t), \\ \partial_x^2\theta(1,t) = a\partial_t\theta(1,t) + b\sqrt{2}\displaystyle\int_{-\infty}^{+\infty}\varphi\,\mathrm{d}\xi, \\ \partial_x^3\theta(1,t) = c\sqrt{2}\displaystyle\int_{-\infty}^{+\infty}(-2\pi\,|\xi|\,\psi + \sqrt{2}\partial_t\theta(1,t))\mathrm{d}\xi + \\ \qquad +d\partial_t\partial_x\theta(1,t). \end{cases} \tag{14}$$

Remark that, in opposite to (13) which has no associated semigroup because of the presence of the time-non local operators $I^{1/2}$ and $\partial_t^{1/2}$, (14) may be expressed under the standard abstract form : $\frac{\mathrm{d}Y}{\mathrm{d}t} = \mathcal{A}Y$, in a convenient Hilbert state space. Although the global system (14) may seem to be more complex than (13), its analysis will be greatly simplified by the presence of the auxiliary state variables φ, ψ.

3 Analysis of the system

For simplicity, we only consider feedbacks defined by (6), with the additional hypothesis : $a = d = 0$. So, the feedback matrix is reduced to :

$$\mathcal{Z} = \begin{bmatrix} 0 & bI^{1/2} \\ c\partial_t^{1/2} & 0 \end{bmatrix},$$

and the analysis of (13) is performed through the equivalent augmented system (14).

We first define energy functionals for the state variables : $\theta, \partial_t\theta, \varphi$ and ψ. The mechanical energy of the beam is classically given by :

$$E_\theta(t) = \frac{1}{2}\int_0^1 \left((\partial_x^2\theta)^2 + (\partial_t\theta)^2\right)\mathrm{d}x, \tag{15}$$

associated to the Hilbert space :

$$\begin{array}{c} \mathcal{H}_b = H(0,1)\times L^2(0,1), \\ H(0,1) = \{h \in L^2(0,1);\ h', h'' \in L^2(0,1), \\ h(0) = h'(0) = 0\}, \end{array} \tag{16}$$

with the scalar product :

$$((e,f)\,|(g,h))_{\mathcal{H}_b} = \int_0^1 (e''g'' + fh)\,\mathrm{d}x.$$

The energies of the diffusive variables φ, ψ are respectively defined by :

$$\begin{array}{l} E_\varphi(t) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{+\infty}\varphi^2\mathrm{d}x, \\ E_\psi(t) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{+\infty}\psi^2\mathrm{d}x, \end{array} \tag{17}$$

associated to the classical Hilbert space $L^2(\mathbf{R})$, with scalar product : $(e\,|f)_{L^2} = \int_{-\infty}^{+\infty} ef\mathrm{d}x$.

We then define the global state vector, energy functional and Hilbert state space respectively :

$$\begin{array}{c} Y(t) := (\theta, \partial_t\theta, \varphi, \psi)^T, \\ E_Y := E_\theta - bE_\varphi + cE_\psi, \\ \mathcal{H} := \mathcal{H}_b \times (L^2(\mathbf{R}))^2. \end{array} \tag{18}$$

The fundamental *a priori* property is obtained [4] :

Proposition 3.1 *Under the assumption $b \le 0$, $c \ge 0$, the global system (14) is dissipative in the sense :*

$$\begin{array}{c} \forall Y(t) \in \mathcal{H},\ \textit{solution of (14)}, \\ E_Y(t) \ge 0, \\ \dfrac{\mathrm{d}E_Y(t)}{\mathrm{d}t} \le 0. \end{array} \tag{19}$$

The proof is straightforward using technical but elementary calculations. From (19), we may deduce by use of classical semigroup theory :

Theorem 3.2 *For any $Y(0) := Y_0 \in \mathcal{H}$, the Cauchy problem (14) admits a unique weak solution in $L^\infty(\mathbf{R}^+;\mathcal{H})$; furthermore, $\forall t \ge 0$, $E_Y(t) \le E_Y(0) = \frac{1}{2}\|Y_0\|^2_{\mathcal{H}}$.*

Corollary 3.3 *If $\varphi_0 = \psi_0 = 0$, then the mechanical energy of the beam satisfies the property :*

$$\forall t \ge 0,\ E_\theta(t) \le E_\theta(0). \tag{20}$$

In order to achieve the stability analysis, we consider the asymptotic behavior of (13).

Under the natural hypothesis : $b < 0$ or $c > 0$, the Lyapunov functional E_Y has the following property [4]:

Proposition 3.4 *If Y is a solution of (14), then* $\left(\dfrac{\mathrm{d}E_Y}{\mathrm{d}t} \equiv 0 \Rightarrow Y \equiv 0\right)$.

It may then be proved that the trajectories $\cup_{t\ge 0}\{(\theta, \partial_t\theta)\}$ are relatively compact in $\mathcal{H}_b$, and so, applying the LaSalle's invariance principle, we conclude : $\|(\theta, \partial_t\theta)\|^2_{\mathcal{H}_b} \underset{t\to+\infty}{\longrightarrow} 0$.

So, under the above assumptions, the system (13) is asymptotically stable. The asymptotic stability of the global system (14) is also obtained, under more technical computations.

Finally, it may be added that, due to the passive nature of the closed-loop system, such controls are robust with respect to uncertainties.

4 Numerical approximation and simulation

In this section, we present numerical simulations of the global system (14) to illustrate both the convenience of the diffusive realizations of the operators $I^{1/2}$ and $\partial_t^{1/2}$ in the context of numerical approximations, and the efficiency of the wave-absorbing feedback controls (8).

4.1 Numerical approximation

More details may be found in [4].

We consider a finite network on the ξ-variable, with convenient hypothesis [4] :

$$\mathcal{N} = \{\xi_1, \xi_2, ..., \xi_K, \xi_{K+1}\}, \ 0 < \xi_k < \xi_{k+1}, \qquad (21)$$

the finite-dimensional differential systems obtained from (10), (11) on $\mathcal{N}$:

$$\dot{\varphi}_k = -4\pi^2\xi_k^2\varphi_k + \sqrt{2}e, \ \varphi_k(0) = 0, \quad k = 1, ..., K, \qquad (22)$$

$$\dot{\psi}_k = -4\pi^2\xi_k^2\psi_k + \sqrt{2}.2\pi\xi_k f, \ \psi_k(0) = 0, \quad k = 1, ..., K, \qquad (23)$$

and the linear interpolations on $\mathcal{N} \cup -\mathcal{N}$ defined by :

$$\begin{cases} \overline{\varphi}(.,t) = \sum_{k=1}^{K} \varphi_k(t)\Lambda_k \\ \overline{\psi}(.,t) = \sum_{k=1}^{K} \psi_k(t)\Lambda_k, \end{cases} \qquad (24)$$

where Λ_k are convenient piecewise affine functions with bounded support (see [4] for details). The functions $\overline{\varphi}, \overline{\psi}$ are classical functional approximations of φ, ψ respectively. The outputs of the systems (22), (23) are respectively defined by :

$$\overline{g}(t) := \sqrt{2}\int \overline{\varphi}\mathrm{d}\xi = \sum_{k=1}^{K} \lambda_k\varphi_k(t), \qquad (25)$$

$$\begin{aligned} \overline{h}(t) &:= \sqrt{2}\int(-2\pi\,|\xi|\,\overline{\psi} + \sqrt{2}f)\mathrm{d}\xi = \\ &= -\sum_{k=1}^{K} \mu_k\psi_k(t) + \lambda f(t), \end{aligned} \qquad (26)$$

$$\begin{aligned} \text{with : } \lambda_k &= \sqrt{2}\int \Lambda_k \mathrm{d}\xi, \\ \mu_k &= \sqrt{2}.2\pi\int |\xi|\,\Lambda_k\mathrm{d}\xi, \\ \lambda &= \sqrt{2}\sum_{k=1}^{K}\lambda_k, \end{aligned} \qquad (27)$$

and we have the following convergence result when the network $\mathcal{N}$ fills up $\mathbf{R}^+$ in a natural way [4] :

Theorem 4.1 *For any $e, f \in L^2(0,T)$,*

$$\begin{cases} \overline{g} \longrightarrow I^{1/2}e & \text{in } L^2(0,T) \\ \overline{h} \longrightarrow \partial_t^{1/2}f & \text{in } H^{-1}(0,T). \end{cases}$$

The proof is based on classical Hilbert methods (see [4]). This last result permits to elaborate efficient numerical simulations of $I^{1/2}e$ and $\partial_t^{1/2}f$ from reduced order differential systems [4].

The following coefficients have been chosen for the numerical simulations :

$$\begin{aligned} K &= 10 \\ \xi_1 &= 0.025 \\ \xi_{10} &= 16 \\ \xi_{k+1} &= r\xi_k \\ r &= 2.0502. \end{aligned}$$

In Fig 1, we can see the magnitude and phasis of the input-output transfer (in the Bode coordinates) of (22),(25). It may be remarked that the approximated transfer function is very close to the ideal one on the pulsation domain $\left[10^{-1}, 10^3\right]$. A similar conclusion concerns (23),(26).

The finite-dimensional approximation of (2) have been performed by classical modal approach (20 eigenmodes) and the global system (14) corresponding to the feedback control defined by (8) is approximated by (22), (23), (24), with the additional coupling defined by (8), (25), (26).

4.2 Results and comments

The following wave-absorbing feedback matrices have been considered :

$$\mathcal{Z}_0 = \begin{bmatrix} -1 & -\sqrt{2}I^{1/2} \\ \sqrt{2}\partial_t^{1/2} & 0 \end{bmatrix},$$

$$\mathcal{Z}_1 = \begin{bmatrix} 0 & -\sqrt{2}I^{1/2} \\ \sqrt{2}\partial_t^{1/2} & 0 \end{bmatrix},$$

$$\mathcal{Z}_2 = \begin{bmatrix} 0 & -\sqrt{2}I^{1/2} \\ 0 & 0 \end{bmatrix}.$$

The simulation time has been limited to $t_{\max} = 0.2$, with $\Delta t = 10^{-4}$. The initial condition for the beam is :

$$\theta_0 = \sum_{k=1}^{15}\frac{1}{k^2}\Phi_k, \ \theta_1 = 0,$$

with Φ_k the eigen-modes of the operator ∂_x^4 defined on the domain $(0,1)$, with null boundary conditions.

In the aim of comparisons, the evolution of the beam deflection in the autonomous case ($u = 0$, $v = 0$) is visible in Fig 2, and reveals complex vibrating behaviors, due to the reflections at the boundaries which are completely reflecting (in this case, the system is conservative).

In Fig 3, 4, we may see the beam deflection in the case of matched boundary conditions defined by the feedback $\mathcal{Z}_0$. Propagations are visible at the beginning of the evolution. They are not reflected at the boundary $x = 1$ whose impedance conditions are transparent by construction : the beam behaves like it would be semi-infinite on the domain $[0, +\infty)$. The transparence property confirms the validity of the diffusive approximations of the operators $I^{1/2}, \partial_t^{1/2}$. Note that for such a result, it has been necessary to extend the frequency band to more than 4 decades, which is very simple from the methodology described above.

In Fig 5, 6, we may see the evolution of the beam deflection corresponding respectively to the feedbacks $\mathcal{Z}_1, \mathcal{Z}_2$. Small reflections do exist.

From the mechanical energy evolution point of vue, we may compare in Fig 7, the respective efficiencies of the considered feedbacks. In the cases of $\mathcal{Z}_1$, in spite of small reflections, the energy absorption efficiency seems approximately equivalent to the non-reflecting case $\mathcal{Z}_0$. In the case of $\mathcal{Z}_2$, the beam energy absorption is more slow, excepted for very short times.

Finally, we may see for illustration in Fig 8, 9, the evolution of the auxiliary diffusive state variables φ_k and ψ_k $(k = 1, \ldots, 10)$, in the case of the non-reflecting feedback $\mathcal{Z}_0$. It may be remarked that at $t \approx 0.155$, there is a significant qualitative change in the behavior of these variables which seem to be no more oscillating after this time. This is possibly due to the fading of the propagating modes, only near-field waves remaining present for long times.

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5 Figures

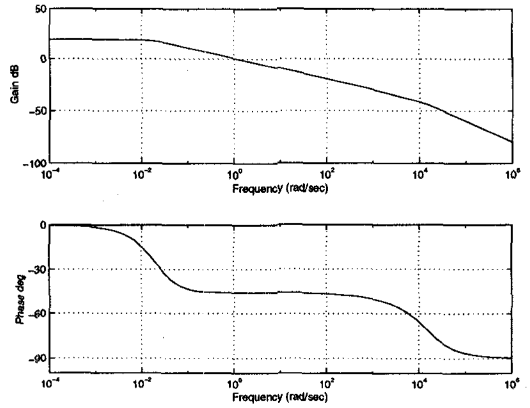

Figure 1: Magnitude and phasis of approximated $I^{1/2}$

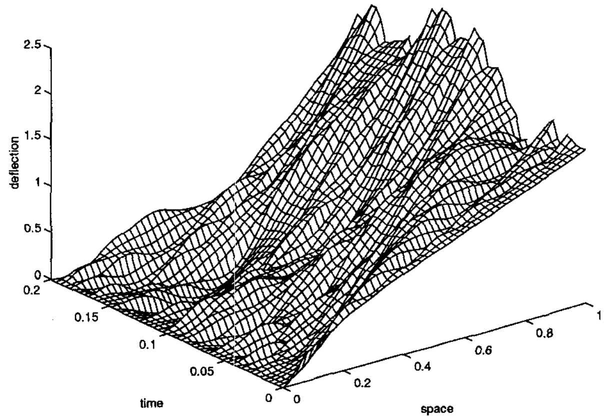

Figure 2: Autonomous beam

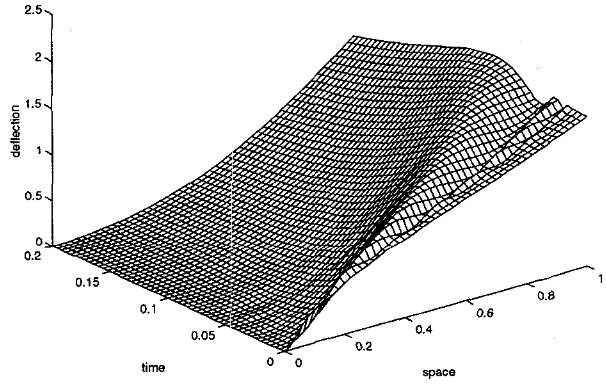

Figure 3: Beam with feedback Z_0

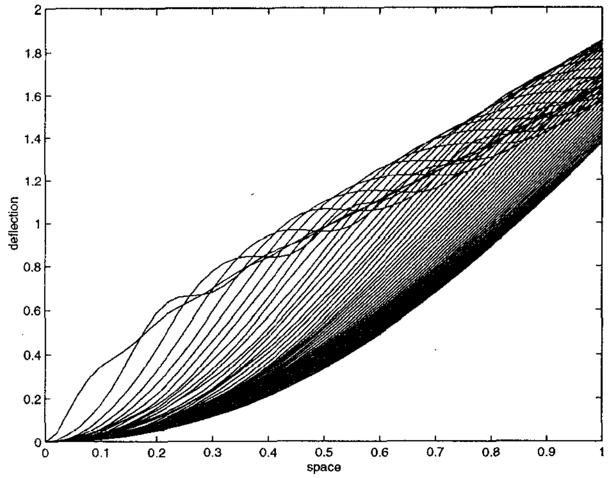

Figure 4: Beam with feedback Z_0

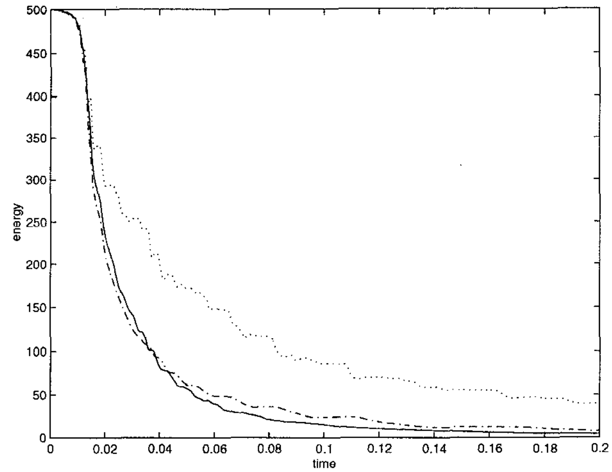

Figure 7: Mechanical energy of the beam : $Z_0(\_)$, $Z_1(\_\_)$, $Z_2(..)$

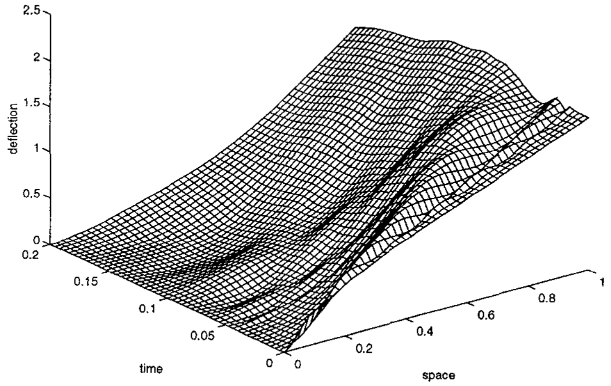

Figure 5: Beam with feedback Z_1

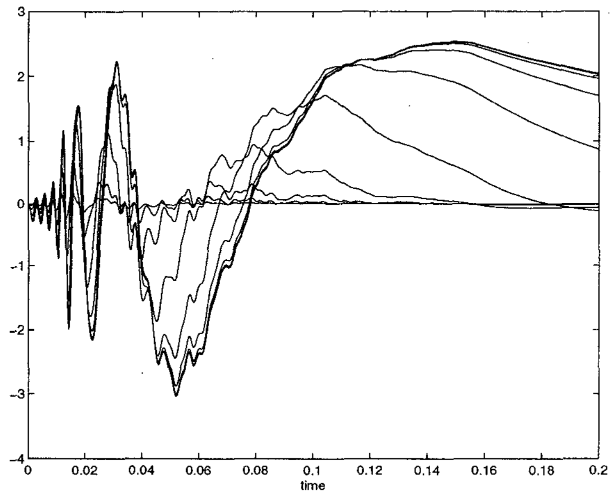

Figure 8: Evolution of $\varphi_k,\ k = 1, 10$

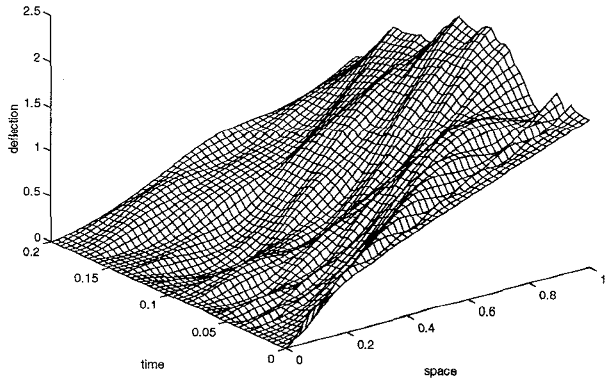

Figure 6: Beam with feedback Z_2

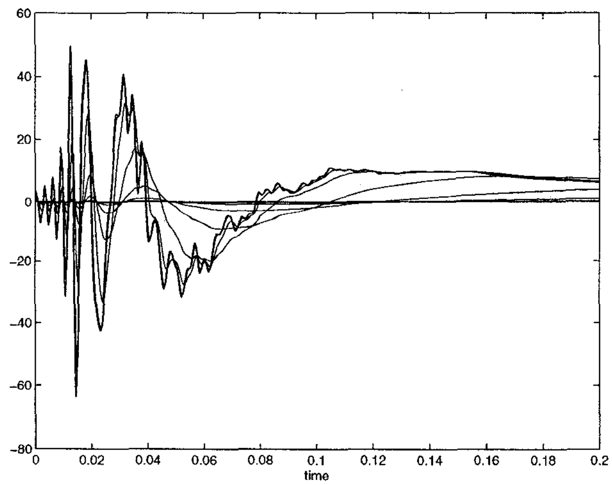

Figure 9: Evolution of $\psi_k,\ k = 1, 10$